10/1 Last time: Derivatives of Multivariable Functions

Directional Derivative: $D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0^+} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h}$

Unit vector in $\mathbb{R}^n$; $\vec{a} \in dom(f)$

n-variable function

<u>Def'n</u>: The k^{th} partial derivative (or the partial derivative w.r.t. x_k) of n-variable function f is $df/dx_k = D_{\vec{e}_k} f$ where

$\vec{e}_k = \langle \underbrace{0, \ldots,}_{\text{all } 0} \underbrace{1,}_{k^{th} \text{ position}} \underbrace{\ldots, 0}_{\text{all } 0} \rangle$

Note: $\vec{e}_K$ is the inc. direction for X_K where $\mathbb{R}^n$ has coordinates $(x_1, x_2, \ldots, x_n)$

What's Going On? - Two variables (x, y). Given functions $f(x, y)$ and $(a, b) \in dom(f)$.

$$\left.\frac{df}{dy}\right|_{(a,b)} = D_{\vec{e}_2} f(a,b)$$

$$= \lim_{h \to 0^+} \frac{f(\langle a, b\rangle + h\vec{e}_2) - f(\langle a, b\rangle)}{h}$$

$$= \lim_{h \to 0^+} \frac{f(a + h \cdot 0, b + h \cdot 1) - f(a, b)}{h}$$

$$= \lim_{h \to 0^+} \frac{f(a, b+h) - f(a, b)}{h}$$ ← first coordinate not changing (held constant)

single variable function

Now let $g(y) := f(a, y)$ an rewrite

Calc I Derivative

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = \lim_{h \to 0^+} \frac{f(a, b+h) - f(a, b)}{h} = \lim_{h \to 0^+} \frac{g(b+h) - g(b)}{h} = g'(b)$$

By construction g treats x as a constant, so g' is the derivative of f "pretending" x is constant.

This works similarly for every component.

Ex: Take all partial derivatives of $f(x,y) = xy^2 - x^{3/2} + \sin(x-y)$

usual derivative (holding y constant)

So all usual rules apply

Sol.: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[xy^2 - x^{3/2} + \sin(x-y)\right]$

$= \frac{\partial}{\partial x}\left[xy^2\right] - \frac{\partial}{\partial x}\left[x^{3/2}\right] + \frac{\partial}{\partial x}\left[\sin(x-y)\right]$

$= y^2 \frac{\partial}{\partial x}\left[x\right] - \frac{3}{2}x^{1/2} + \cos(x-y)\frac{\partial}{\partial x}\left[x-y\right]$

$= y^2 - \frac{3}{2}x^{1/2} + \cos(x-y)$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[xy^2 - x^{3/2} + \sin(x-y)\right]$

$= \frac{\partial}{\partial y}\left[xy^2\right] - \frac{\partial}{\partial y}\left[x^{3/2}\right] + \frac{\partial}{\partial y}\left[\sin(x-y)\right]$

$= x\frac{\partial}{\partial y}\left[y^2\right] - 0 + \cos(x-y)\frac{\partial}{\partial y}\left[x-y\right]$

$= x(2y) + \cos(x-y)(-1)$

$= 2xy - \cos(x-y)$

Ex: Compute partial derivatives of $f(x,y,z) = e^{x^2+y^2}\sin(xz)\cos(yz)$

Sol: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right]$

$= \cos(yz)\frac{\partial}{\partial x}\left[e^{x^2+y^2}\sin(xz)\right]$

$= \cos(yz)\left(\frac{\partial}{\partial x}\left[e^{x^2+y^2}\right]\sin(xz) + e^{x^2+y^2}\frac{\partial}{\partial x}\left[\sin(xz)\right]\right)$

$= \cos(yz)\left(2xe^{x^2+y^2}\sin(xz) + e^{x^2+y^2}\cos(xz)z\right)$

$= \cos(yz)e^{x^2+y^2}\left(2x\sin(xz) + z\cos(xz)\right)$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right]$$

$$= \sin(xz)\frac{\partial}{\partial y}\left[e^{x^2+y^2}\cos(yz)\right] = \sin(xz)\left(\frac{\partial}{\partial y}\left[e^{x^2+y^2}\right]\cos(yz) + e^{x^2+y^2}\frac{\partial}{\partial y}\left[\cos(yz)\right]\right)$$

$$= \sin(xz)\left(2y\,e^{x^2+y^2}\cos(yz) + e^{x^2+y^2}\left(-\sin(yz)z\right)\right)$$

$$= e^{x^2+y^2}\sin(xz)\left(2y\cos(yz) - z\sin(yz)\right)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[e^{x^2+y^2}\sin(xz)\cos(yz)\right] = e^{x^2+y^2}\frac{\partial}{\partial z}\left[\sin(xz)\cos(yz)\right]$$

$$= e^{x^2+y^2}\left(\frac{\partial}{\partial z}\left[\sin(xz)\right]\cos(yz) + \sin(xz)\frac{\partial}{\partial z}\left[\cos(yz)\right]\right)$$

$$= e^{x^2+y^2}\left(x\cos(xz)\cos(yz) + \sin(xz)(-y\sin(yz))\right)$$

$$= e^{x^2+y^2}\left(x\cos(xz)\cos(yz) - y\sin(xz)\sin(yz)\right)$$

Note: everything w/ partial derivatives is working out mostly the same as calc I, once we hold variables constant

We can make second order derivatives in exactly the same way as we did in calc I — Now there's just 1 more

↳ "derivative of derivative"

$$\underbrace{\frac{\partial^2 f}{(\partial x)^2},\ \frac{\partial^2 f}{(\partial y)^2}}_{\text{"pure second order" partials}},\quad \underbrace{\frac{\partial^2 f}{\partial y\,\partial x},\ \frac{\partial^2 f}{\partial x\,\partial y}}_{\text{"mixed second order" partials}}$$

Ex: compute second order partials of $f(x,y) = xy^2 - x^{3/2} + \sin(x-y)$

Sol. We computed earlier:

$\partial f/\partial x = y^2 - \frac{3}{2}x^{1/2} + \cos(x-y)$ and $\partial f/\partial y = 2xy - \cos(x-y)$

Now we compute

$$\partial^2 f/(\partial x)^2 = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left[y^2 - 3/2\, x^{1/2} + \cos(x-y)\right] = -\frac{3}{4}x^{-1/2} - \sin(x-y)$$

$$\partial^2 f/(\partial y)^2 = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}\left[2xy - \cos(x-y)\right] = 2x - \sin(x-y)$$

Mixed partials:

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}\left[y^2 - 3/2\, x^{1/2} + \cos(x-y)\right]$$
$$= 2y - 0 - \sin(x-y)(-1) = 2y + \sin(x-y)$$

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[2xy - \cos(x-y)\right] = 2y - (-\sin(x-y)\cdot 1) = 2y + \sin(x-y)$$

Note: Up to this point, applying partial derivatives just works in exactly the same way as calc I

Want: understand mixed partial derivatives...

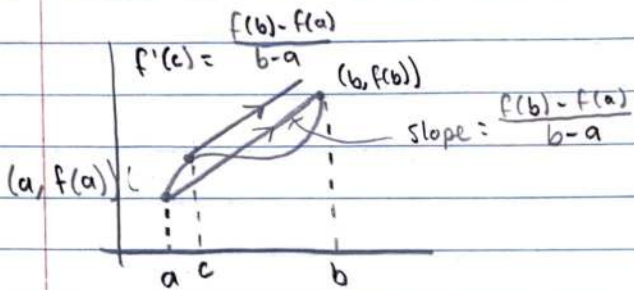
1) Why did the previous example have $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$?

2) How do we guarantee (or tell in advance) if this happens for future functions?

To answer these questions, we need to recall some calc I:

Mean Value Theorem (MVT): Let $f(t)$ be a function which is diff on (a, b) and cts on $[a, b]$

Then there is $a < c < b$ such that $f'(c)(b-a) = f(b) - f(a)$



Assume the mixed partials question uses MVT

Clairaut's Theorem: Let $f(x, y)$ have cts second-order mixed partial derivatives on a disk containing (a, b). Then at (a, b) we have

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(a,b)} = \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(a,b)}$$